

# CONORMAL MODULES VIA PRIMITIVE IDEALS

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**ABSTRACT.** The main object of this note is to study the conormal module  $M$  and the computation of the second symbolic power  $\bar{\mathfrak{g}}^{(2)}$  of an ideal  $\bar{\mathfrak{g}}$  in the residue ring  $\mathcal{O}/\mathfrak{h}$  of a polynomial ring  $\mathcal{O}$  over a field of characteristic zero. The torsion part  $T(M)$  of  $M$  and the torsion free module  $M/T(M)$  are expressed by the primitive ideal of  $\mathfrak{g}$  relative to  $\mathfrak{h}$ . Two characterizations for  $M/T(M)$  to be free are proved. Some immediate applications are worked out.

## 1. INTRODUCTION

Let  $\mathfrak{h} \subset \mathfrak{g}$  be two ideals of a polynomial ring  $\mathcal{O}$  over a field  $k$  of characteristic zero. Let  $\bar{\mathfrak{g}} := \mathfrak{g}/\mathfrak{h}$  be the image of  $\mathfrak{g}$  in  $\mathcal{O}/\mathfrak{h}$  under the canonical projection. The main interest of this article is to study the conormal module  $M := \bar{\mathfrak{g}}/\bar{\mathfrak{g}}^2$  and the second symbolic power  $\bar{\mathfrak{g}}^{(2)}$  of  $\bar{\mathfrak{g}}$ . The connection between  $M$  and  $\bar{\mathfrak{g}}^{(2)}$  is established by the primitive ideal of  $\mathfrak{g}$  (relative to  $\mathfrak{h}$ ), which was introduced by Siersma-Pellikaan [10, 11] and generalized to relative version in [5, 6].

In general,  $\mathcal{O}/\mathfrak{h}$  is not regular, so the  $\mathcal{O}/\mathfrak{g}$  module  $M$  is neither free nor torsion free even if both  $\mathfrak{h}$  and  $\mathfrak{g}$  are complete intersections, and the projective dimension of  $\bar{\mathfrak{g}}$  is not finite. Especially, no generating set of  $\bar{\mathfrak{g}}$  forms a regular sequence. Then the following questions would be interesting.

- a) Find descriptions of the torsion part  $T(M)$  of  $M$ , calculate the length (when it is finite) of  $T(M)$ ;
- b) Find descriptions of the torsion free module  $N := M/T(M)$  and conditions on the freeness of  $N$ .

In commutative algebra, there is a question by Vasconcelos on how to compute effectively the symbolic powers of an ideal in a residue ring of a polynomial ring [14, 16]. It follows from [6] that the second symbolic power of  $\bar{\mathfrak{g}}$  is the image of the primitive ideal of  $\mathfrak{g}$  in the residue ring. We give a precise expression of  $\bar{\mathfrak{g}}^{(2)}$  under some assumptions.

What brought our attention to these questions is the studying of functions with non-isolated singularities on singular spaces. In general the ideals  $\mathfrak{h} \subset \mathfrak{g}$  define two subvarieties  $X \supset \Sigma$  of  $\mathbb{C}^n$  if  $k = \mathbb{C}$ . The primitive ideal of  $\mathfrak{g}$  collects all the functions whose zero level hypersurfaces pass through  $\Sigma$  and are tangent to the regular part  $X_{\text{reg}}$  of  $X$  along  $\Sigma \cap X_{\text{reg}}$ . If we supply  $X$  with the so called *logarithmic stratification* [12], then the primitive ideal of  $\mathfrak{g}$  consists of exactly all the functions from  $\mathfrak{g}$  whose stratified critical loci on  $X$  contain  $\Sigma$  (cf. [5]). Hence, locally the primitive ideal plays a similar role to the second power of the maximal ideal of the local ring  $\mathcal{O}_{\mathbb{C}^n, 0}$  in singularity theory. In order to study the topology of the Milnor fibre  $F_f$  of a function

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$f$  with singular locus  $\Sigma$ , we use a good deformation (the Morsification)  $f_s$  of  $f$ . This  $f_s$  has relatively simpler singularities than  $f$ . The existence of the good deformation and related invariants (both topological and algebraic) have close relationship with  $M$ ,  $T(M)$  and  $N$ . Roughly speaking, the freeness of  $N$  implies the existence of the good deformation [5]. The length of torsion module  $T(M)$  (when it is finite) gives some information on how  $\Sigma$  sits in  $X$  (cf. [7]).

Under some conditions, we answer the questions a) and b). More precisely, after some descriptions of  $T(M)$  and  $N$ , we mainly prove the following (see also Remark 12)

**Main Theorem** *Let  $\mathfrak{h} \subset \mathfrak{g}$  be complete intersection ideals of a polynomial ring  $\mathcal{O}$  over a field  $k$  of characteristic zero. Let  $\text{Spec}(\mathcal{O}/\mathfrak{g})$  be reduced and connected, and the Jacobian ideal  $\mathcal{J}(\mathfrak{h})$  of  $\mathfrak{h}$  be not contained in any minimal prime of  $\mathcal{O}/\mathfrak{g}$ . The  $\mathcal{O}/\mathfrak{g}$ -module  $N$  is free if and only if there exists an  $\mathcal{O}$ -regular sequence  $g_1, \dots, g_n$  generating  $\mathfrak{g}$ , such that*

$$\int_{\mathfrak{h}} \mathfrak{g} = (g_1, \dots, g_p) + (g_{p+1}, \dots, g_n)^2,$$

where  $p := \text{grade } \mathfrak{h}$ ,  $n := \text{grade } \mathfrak{g}$ .

As an application, in the last section we study lines on a variety with isolated complete intersection singularity. More applications to the general deformation theory of non-isolated singularities on singular spaces will be given in the sequel papers.

## 2. PRIMITIVE IDEALS

Let  $\mathcal{O}$  be a commutative ring and  $k \subset \mathcal{O}$  be a subring. Let  $\text{Der}(\mathcal{O})$  denote the  $\mathcal{O}$ -module of all the  $k$ -derivations of  $\mathcal{O}$ . For an ideal  $\mathfrak{h} \subset \mathcal{O}$ , define

$$\text{Der}_{\mathfrak{h}}(\mathcal{O}) := \{\xi \in \text{Der}(\mathcal{O}) \mid \xi(\mathfrak{h}) \subset \mathfrak{h}\}.$$

**Definition 1.** Let  $\mathfrak{h} \subset \mathfrak{g} \subset \mathcal{O}$  be ideals. The *primitive ideal of  $\mathfrak{g}$  relative to  $\mathfrak{h}$*  is

$$\int_{\mathfrak{h}} \mathfrak{g} := \{f \in \mathfrak{g} \mid \xi(f) \in \mathfrak{g} \text{ for any } \xi \in \text{Der}_{\mathfrak{h}}(\mathcal{O})\}.$$

*Remark 2.* This definition is a generalization of [10, 11], and was generalized to higher order relative version in [6]. And under general assumptions (cf. Proposition 4), it was proved in [6] that the primitive ideal of  $\mathfrak{g}$  relative to  $\mathfrak{h}$  is the inverse image in  $\mathcal{O}$  of the second symbolic power of the quotient ideal  $\mathfrak{g}/\mathfrak{h} \subset \mathcal{O}/\mathfrak{h}$ .

Some basic facts about the primitive ideals are collected in the following lemma.

**Lemma 3.** *Let  $\mathfrak{h} \subset \mathfrak{g}$  be ideals of  $\mathcal{O}$ , a commutative noetherian  $k$ -algebra. Then*

- (1)  $\int_{\mathfrak{h}} \mathfrak{g}$  is contained in  $\mathfrak{g}$  and contains  $\mathfrak{h}$  and  $\mathfrak{g}^2$ ;
- (2) for any  $\mathfrak{g}_i \supset \mathfrak{h}$  ( $i = 1, 2$ ),  $\int_{\mathfrak{h}} \mathfrak{g}_1 \cap \mathfrak{g}_2 = \int_{\mathfrak{h}} \mathfrak{g}_1 \cap \int_{\mathfrak{h}} \mathfrak{g}_2$ ;
- (3) If  $\mathcal{O}$  is a polynomial ring over a field of characteristic zero,  $(\int_{\mathfrak{h}} \mathfrak{g})/\mathfrak{h} = \int_0(\mathfrak{g}/\mathfrak{h})$ , where  $0 = \mathfrak{h}/\mathfrak{h}$ ;
- (4) If  $\mathcal{O}$  is a  $k$ -algebra of finite type over a commutative ring  $k$ , and  $\mathfrak{h}$  and  $\mathfrak{g}$  have no embedded primes, then for any multiplicative set  $\mathcal{S} \subset \mathcal{O}$ ,  $\mathcal{S}^{-1}(\int_{\mathfrak{h}} \mathfrak{g}) \simeq \int_{\mathcal{S}^{-1}\mathfrak{h}} \mathcal{S}^{-1}\mathfrak{g}$ .

*Proof.* The first three statements follows immediately from the definition. To prove (4), one may use the fact that  $\mathcal{S}^{-1}\text{Der}_{\mathfrak{h}}(\mathcal{O}) \simeq \text{Der}_{\mathcal{S}^{-1}\mathfrak{h}}(\mathcal{S}^{-1}\mathcal{O})$ . See [6] for details.  $\square$

## 3. CONORMAL MODULE: THE TORSION PART

Let  $M$  be a module over a ring  $\mathcal{O}$ , and  $S$  be the set of all the non-zero divisors of  $\mathcal{O}$ . Denote by  $Q$  the total quotient ring of  $\mathcal{O}$  with denominator set  $S$ , and by  $M_S := M \otimes Q$  the quotient module of  $M$ . The kernel of the canonical map  $M \rightarrow M_S$  is denoted by  $T(M)$  and is called the torsion (part) of  $M$ .

Let  $\mathfrak{h} \subset \mathfrak{g}$  be ideals of the commutative noetherian ring  $\mathcal{O}$ . Denote by  $\bar{\mathfrak{g}} := \mathfrak{g}/\mathfrak{h}$ , the quotient ideal in  $\mathcal{O}/\mathfrak{h}$ . The  $\mathcal{O}/\mathfrak{g}$ -module  $M := \bar{\mathfrak{g}}/\bar{\mathfrak{g}}^2 \simeq \mathfrak{g}/\mathfrak{g}^2 + \mathfrak{h}$  is called the conormal module of  $\bar{\mathfrak{g}}$ .

**Proposition 4.** *Let  $\mathcal{O}$  be a polynomial ring over a field  $k$  of characteristic 0. Let  $\mathfrak{h} \subset \mathfrak{g}$  be unmixed ideals of  $\mathcal{O}$  with  $\mathfrak{g}$  radical, such that the Jacobian ideal  $\mathcal{J}(\mathfrak{h})$  of  $\mathfrak{h}$  is not contained in any minimal prime of  $\mathcal{O}/\mathfrak{g}$ , then*

$$T(M) = T := \frac{\int_{\mathfrak{h}} \mathfrak{g}}{\mathfrak{g}^2 + \mathfrak{h}} \simeq \frac{\bar{\mathfrak{g}}^{(2)}}{\bar{\mathfrak{g}}^2}.$$

Consequently, we have the following exact sequence

$$0 \longrightarrow T(M) \xrightarrow{\iota} M \xrightarrow{\pi} N \longrightarrow 0 \quad (3.1)$$

where  $N := \mathfrak{g}/\int_{\mathfrak{h}} \mathfrak{g}$ .

*Proof.* In the following we use  $\bar{\mathfrak{p}}$  to denote the image of an ideal  $\mathfrak{p}$  of  $\mathcal{O}$  in the residue ring  $\mathcal{O}/\mathfrak{h}$ . We first prove that  $T \subset T(M)$ . Note that for any  $\mathfrak{p} \in \mathcal{V}(\mathfrak{g}) \setminus (\mathcal{V}(\mathcal{J}(\mathfrak{h})) \cup \mathcal{V}(\mathcal{J}(\mathfrak{g})))$ ,  $T_{\mathfrak{p}} = (\int_0 \bar{\mathfrak{g}}_{\bar{\mathfrak{p}}})/\bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}^2$  is a module over local ring  $R := (\mathcal{O}/\mathfrak{h})_{\bar{\mathfrak{p}}}$ . Since  $R$  and  $R/\bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}$  are regular,  $\bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}$  is generated just by a part of the regular system of parameters  $\{g_1, \dots, g_d\}$ . (cf. [9, Theorem 36]). We may also assume that  $\mathfrak{g}$  is prime. It follows from [4] that  $\bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}^2 = \bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}^{(2)}$ , the second symbolic power of  $\bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}$ . By [13],  $\int_0 \bar{\mathfrak{g}}_{\bar{\mathfrak{p}}} = \bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}^{(2)}$ . Hence  $T$  is annihilated by a power of  $\mathcal{J}(\mathfrak{g})\mathcal{J}(\mathfrak{h})$ .

For any  $\bar{a} \in T(M)$ , let  $\bar{\beta} \in \mathcal{O}/\mathfrak{g}$  be a non-zero divisor such that  $\bar{\beta}\bar{a} = \bar{0}$ . By taking representative, we have  $\beta a \in \mathfrak{g}^2 + \mathfrak{h}$ . For any  $\xi \in \text{Der}_{\mathfrak{h}}(\mathcal{O})$ , we have  $\xi(\beta a) \equiv \beta \xi(a) \pmod{\mathfrak{g}}$ . Since  $\xi(\beta a) \in \mathfrak{g}$ , hence  $\xi(a) \in \mathfrak{g}$ , and  $a \in \int_{\mathfrak{h}} \mathfrak{g}$ .

By [6],  $(\int_{\mathfrak{h}} \mathfrak{g})/\mathfrak{h} = \bar{\mathfrak{g}}^{(2)}$ , hence  $T \simeq \bar{\mathfrak{g}}^{(2)}/\bar{\mathfrak{g}}^2$ .  $\square$

**Lemma 5.** *Let  $\mathfrak{g}$  be a radical ideal of a commutative noetherian ring  $\mathcal{O}$ . Suppose that  $\mathfrak{g}$  is generated by an  $\mathcal{O}$ -regular sequence  $g_1, \dots, g_n$ . If  $\mathfrak{g}^* := (g_1, \dots, g_t) + (g_{t+1}, \dots, g_n)^2$  for some  $1 \leq t \leq n$ , then  $\text{Ass}(\mathcal{O}/\mathfrak{g}^*) = \text{Ass}(\mathcal{O}/\mathfrak{g})$ .*

*Proof.* We use the fact: For two ideals  $J \subset I$  with  $I$  radical and  $\sqrt{J} = I$ , then  $\text{Ass}(\mathcal{O}/J) = \text{Ass}(I/J) \cup \text{Ass}(\mathcal{O}/I)$ .

Since  $\sqrt{\mathfrak{g}^*} = \mathfrak{g}$ , we need to prove that  $\text{Ass}(\mathfrak{g}/\mathfrak{g}^*) \subset \text{Ass}(\mathcal{O}/\mathfrak{g})$ . Let  $\mathfrak{g} = \bigcap_{i=1}^r P_i$  be the unique minimal prime decomposition of  $\mathfrak{g}$ . Let  $Q \in \text{Ass}(\mathfrak{g}/\mathfrak{g}^*)$ . By definition, there exist  $0 \neq \bar{x}_i \in \mathcal{O}/\mathfrak{g}$  and  $0 \neq \bar{y} \in \mathfrak{g}/\mathfrak{g}^*$  such that  $P_i = \text{Ann}(\bar{x}_i)$  ( $i = 1, \dots, r$ ) and  $Q = \text{Ann}(\bar{y})$ . Suppose that  $q \in Q \setminus \bigcup_{i=1}^r P_i \neq \emptyset$ . Then  $qy \in \mathfrak{g}^*$ , and  $qx_i \notin \mathfrak{g}$  ( $i = 1, \dots, r$ ). Write  $qy \equiv \beta_1 g_1 + \dots + \beta_t g_t \pmod{\mathfrak{g}^2}$  and  $y = \sum_{k=1}^n y_k g_k$ . Then  $qy - (\beta_1 g_1 + \dots + \beta_t g_t) = (qy_1 - \beta_1)g_1 + \dots + (qy_t - \beta_t)g_t + qy_{t+1}g_{t+1} + \dots + qy_n g_n \equiv 0$

mod  $\mathfrak{g}^2$ . It follows that  $qy_{t+j} \in \mathfrak{g} = \bigcap_{i=1}^r P_i$ . Since  $q \notin P_i$  for all  $i = 1, \dots, r$ ,  $y_{t+j} \in \bigcap_{i=1}^r P_i = \mathfrak{g}$ , which implies that  $y \in \mathfrak{g}^*$ , a contradiction. We may assume that  $Q \subset P_1$ .

On the other hand, since  $\text{Ass}(\mathfrak{g}/\mathfrak{g}^*) \subset \text{Ass}(\mathcal{O}/\mathfrak{g}^*)$  and  $\text{rad}(\mathfrak{g}^*) = \mathfrak{g}$ , we have  $Q \supset \mathfrak{g} = \bigcap_{i=1}^r P_i$ . It follows from [1, (1.11)] that  $Q \supset P_i$  for some  $1 \leq i \leq r$ , which implies that  $i = 1$  since  $\mathfrak{g} = \bigcap_{i=1}^r P_i$  is the minimal prime decomposition of  $\mathfrak{g}$ .  $\square$

**Proposition 6.** *Let  $\mathfrak{h} \subset \mathfrak{g}$  be ideals of  $\mathcal{O}$ , a polynomial ring over a field  $k$  of characteristic zero. Assume that  $\mathfrak{h}$  is unmixed and  $\mathfrak{g}$  is a radical complete intersection ideal. Suppose that the Jacobian ideal  $\mathcal{J}(\mathfrak{h})$  of  $\mathfrak{h}$  is not contained in any minimal prime of  $\mathcal{O}/\mathfrak{g}$ . If there exists a minimal generating set  $\{g_1, \dots, g_n\}$  of  $\mathfrak{g}$  such that*

- (1) *The images of  $g_1, \dots, g_t$  are contained in  $T(M)$  for some integer  $t : 0 \leq t \leq n$ ;*
- (2) *For any prime  $\mathfrak{p} \in \text{Ass}(\mathcal{O}/\mathfrak{g})$ , the images of  $g_{t+1}, \dots, g_n$  in  $(\mathcal{O}/\mathfrak{h})_{\bar{\mathfrak{p}}}$  generate  $(\mathfrak{g}/\mathfrak{h})_{\bar{\mathfrak{p}}}$  as a complete intersection ideal of  $(\mathcal{O}/\mathfrak{h})_{\bar{\mathfrak{p}}}$ .*

*Then  $\int_{\mathfrak{h}} \mathfrak{g} = \mathfrak{g}^* := (g_1, \dots, g_t) + (g_{t+1}, \dots, g_n)^2$ .*

*Proof.* It is obvious that  $\mathfrak{g}^* \subset \int_{\mathfrak{h}} \mathfrak{g}$ . It was proved in [6] that  $\mathcal{O}/\mathfrak{g}$  and  $\mathcal{O}/\int_{\mathfrak{h}} \mathfrak{g}$  have the same associated ideals. By Lemma 3 and 5, neither  $\mathfrak{g}^*$  nor  $\int_{\mathfrak{h}} \mathfrak{g}$  has embedded primes and these two ideals have common radical  $\mathfrak{g}$ . We only need to prove they are equal locally at any associated prime  $\mathfrak{p}$  of  $\mathcal{O}/\mathfrak{g}$ .

Let  $f = a_1g_1 + \dots + a_ng_n \in \int_{\mathfrak{h}} \mathfrak{g}$ . Then for any  $\xi \in \text{Der}_{\mathfrak{h}}(\mathcal{O})$ ,  $\xi(f) \equiv \xi(f') \equiv 0 \pmod{\mathfrak{g}}$ , where  $f' := a_{t+1}g_{t+1} + \dots + a_ng_n$ . By assumption  $\mathfrak{p} \in \mathcal{V}(\mathfrak{h}) \setminus \mathcal{V}(\mathcal{J}(\mathfrak{h}))$ ,  $\tilde{\xi}(\tilde{f}')$  is contained in  $(\mathfrak{g}/\mathfrak{h})_{\bar{\mathfrak{p}}}$ , where we use  $\tilde{e}$  to denote the image of an element  $e \in \mathcal{O}$  (resp.  $\text{Der}_{\mathfrak{h}}(\mathcal{O})$ ) under taking modulo  $\mathfrak{h}$  and localization at  $\bar{\mathfrak{p}}$ . In other words,

$$\tilde{f}' = \tilde{a}_{t+1}\tilde{g}_{t+1} + \dots + \tilde{a}_n\tilde{g}_n \in \left( \left( \int_{\mathfrak{h}} \mathfrak{g} \right) / \mathfrak{h} \right)_{\bar{\mathfrak{p}}} = \int_0 (\mathfrak{g}/\mathfrak{h})_{\bar{\mathfrak{p}}} = (\tilde{g}_{t+1}, \dots, \tilde{g}_n)_{\bar{\mathfrak{p}}}^2,$$

where the last equality follows from Lemma 3 and [13]. By the assumption (2), we have  $((a_{t+1}, \dots, a_n)/\mathfrak{h})_{\bar{\mathfrak{p}}} \subset ((g_{t+1}, \dots, g_n)/\mathfrak{h})_{\bar{\mathfrak{p}}}$ , which implies that  $(a_{t+1}, \dots, a_n)_{\mathfrak{p}} \subset \mathfrak{g}_{\mathfrak{p}}$ . Thus, we proved that  $(\mathfrak{g}^*)_{\mathfrak{p}} = \left( \int_{\mathfrak{h}} \mathfrak{g} \right)_{\mathfrak{p}}$ .  $\square$

**Corollary 7.** *Under the assumption of the Proposition 6,  $N$  is a free  $\mathcal{O}/\mathfrak{g}$ -module with the images  $\hat{g}_{t+1}, \dots, \hat{g}_n$  of  $g_{t+1}, \dots, g_n$  as basis, and  $M \simeq T(M) \oplus N$ .*

*Proof.* We only need to prove that  $\hat{g}_{t+1}, \dots, \hat{g}_n$  are  $\mathcal{O}/\mathfrak{g}$ -linearly independent. Suppose there are some  $\bar{\beta}_{t+j} \in \mathcal{O}/\mathfrak{g}$  such that  $\hat{a} := \bar{\beta}_{t+1}\hat{g}_{t+1} + \dots + \bar{\beta}_n\hat{g}_n = 0$  in  $N$ . By the expression of the primitive ideal in Proposition 6 and taking representatives, we have  $a \in (g_{t+1}, \dots, g_n) \cap \int_{\mathfrak{h}} \mathfrak{g} = (g_1, \dots, g_t) \cap (g_{t+1}, \dots, g_n) + (g_{t+1}, \dots, g_n)^2$ . Let  $a \equiv \beta_1g_1 + \dots + \beta_tg_t \pmod{(g_{t+1}, \dots, g_n)^2}$ . Then  $-\beta_1g_1 - \dots - \beta_tg_t + \beta_{t+1}g_{t+1} + \dots + \beta_ng_n \equiv 0 \pmod{\mathfrak{g}^2}$ , from which and the assumption on  $\mathfrak{g}$ , it follows that  $\beta_j \in \mathfrak{g}$ .  $\square$

Let  $\mathfrak{g}$  be generated by an  $\mathcal{O}$ -regular sequence:  $g_1, \dots, g_n$ . Assume that there exist an integer  $0 \leq t \leq n$  and non-zero divisors  $\bar{\beta}_1, \dots, \bar{\beta}_t \in \mathcal{O}/\mathfrak{g}$  such that  $\bar{\beta}_1\hat{g}_1, \dots, \bar{\beta}_t\hat{g}_t$

are zero in  $M$  as an  $\mathcal{O}/\mathfrak{g}$ -module. Namely  $\beta_1 g_1, \dots, \beta_t g_t \in \mathfrak{g}^2 + \mathfrak{h}$ , where  $\beta_i g_i$  is the representative of  $\bar{\beta}_i \hat{g}_i$ .

Let  $\{h_1, \dots, h_p\}$  be a minimum generating set of  $\mathfrak{h}$ . Denote

$$h = {}^T(h_1, \dots, h_p), \quad g = {}^T(g_1, \dots, g_n), \quad G = {}^T(G_1, \dots, G_t), \quad \Lambda = \text{diag}\{\beta_1, \dots, \beta_t\},$$

where  $T$  means the transposition of the matrix indicated.

Let  $A$  and  $B = (B_1 : B_2)$  be the matrices such that  $\Lambda {}^T(g_1, \dots, g_t) = Ah + G$ ,  $h = Bg$ , where  $A$  is a  $t \times p$  matrix,  $B$  a  $p \times n$  matrix,  $B_1$  a  $p \times t$  matrix,  $B_2$  a  $p \times (n - t)$  matrix, and  $G_i \in \mathfrak{g}^2$ . Let  $C_1 = AB_1, C_2 = AB_2$ , then we have

$$(\Lambda - C_1) {}^T(g_1, \dots, g_t) - C_2 {}^T(g_{t+1}, \dots, g_n) \equiv 0 \pmod{\mathfrak{g}^2}$$

Note that  $\mathfrak{g}/\mathfrak{g}^2$  is a free  $\mathcal{O}/\mathfrak{g}$ -module. We have  $\bar{\Lambda} = \bar{C}_1, \bar{C}_2 = 0$  in  $\mathcal{O}/\mathfrak{g}$ . From this we obtain the following lemma similar to the implicit function theorem.

**Lemma 8.** *Let  $\mathfrak{g}$  be a radical complete intersection ideal of  $\mathcal{O}$ , then  $\det \bar{C}_1 = \det \bar{\Lambda} = \bar{\beta}_1 \cdots \bar{\beta}_t$  is a non-zero divisor in  $\mathcal{O}/\mathfrak{g}$  and consequently  $\text{rank}(B_1) \geq t$  and  $t \leq p$ .  $\square$*

#### 4. FREENESS OF $N$ AND THE PRIMITIVE IDEAL

Let  $\mathfrak{h} \subset \mathfrak{g}$  be complete intersection ideals of  $\mathcal{O} = k[x_0, x_1, \dots, x_m]$ , a polynomial ring over a field  $k$  of characteristic zero. Assume that the Jacobian  $\mathcal{J}(\mathfrak{h})$  is not contained in any minimal prime of  $\mathcal{O}/\mathfrak{g}$ . Let  $p := \text{grade } \mathfrak{h}, n := \text{grade } \mathfrak{g}$ . If  $\mathfrak{g}$  is radical and generated by an  $\mathcal{O}$ -regular sequence  $g_1, \dots, g_n$ , we can choose the minimal generating set  $\{h_1, \dots, h_p\}$  of  $\mathfrak{h}$  such that (with changing of the generators of  $\mathfrak{g}$  if necessary):

$$h_i \equiv \sum_{j=1}^t b_{ij} g_j \pmod{\mathfrak{g}^2}, \quad 1 \leq i \leq p \quad (4.1)$$

where  $t$  is an integer  $0 \leq t \leq n$ ,  $b_{ij} \notin \mathfrak{g} \setminus 0$ , and for each  $j$ ,  $(b_{1j}, \dots, b_{pj}) \neq 0$  in  $(\mathcal{O}/\mathfrak{g})^p$ , (otherwise one could lower  $t$ ). Denote  $B := (b_{ij})$ .

**Lemma 9.** *Under the assumptions above, we have*

- (1)  $t \geq p$ ;
- (2) *There exists at least one maximal minor of  $B$  which is non-zero divisor in  $\mathcal{O}/\mathfrak{g}$ .*

*Proof.* Since  $\mathfrak{h}$  is a complete intersection, the Jacobian  $\mathcal{J}(\mathfrak{h})$  of  $\mathfrak{h}$  can be generated by  $\mathfrak{h}$  and the  $p \times p$  minors of the Jacobian matrix  $J(\mathfrak{h})$  of  $\mathfrak{h}$ . Since each of these minors, say  $\Delta_{j_1, \dots, j_p}$ , is the determinant of  $BG_{j_1, \dots, j_p}$  modulo  $\mathfrak{g}$ , where  $G_{j_1, \dots, j_p}$  is the  $t \times p$  submatrix of  $J(\mathfrak{g})$ , consisting of the  $0 \leq j_1 < \dots < j_p \leq n$  columns of  $J(\mathfrak{g})$ , the Jacobian matrix of  $\mathfrak{g}$ . Suppose  $t < p$ , there would be,  $\det(BG_{j_1, \dots, j_p}) \equiv 0 \pmod{\mathfrak{g}}$ . This is impossible since we assume that  $\mathcal{J}(\mathfrak{h})$  is not contained in any minimal prime of  $\mathcal{O}/\mathfrak{g}$ . This proves (1).

(2) Suppose that all the  $p \times p$  minors of  $B$  are zero divisor in  $\mathcal{O}/\mathfrak{g}$ . Then there exists  $0 \neq a \in \mathcal{O}/\mathfrak{g}$  such that  $ab_1 \wedge b_2 \wedge \dots \wedge b_p = 0$  in  $\bigwedge^p (\mathcal{O}/\mathfrak{g})^t$ , where  $b_i \in (\mathcal{O}/\mathfrak{g})^t$  is the image of the  $i$ -th row vector of  $B$ . Hence  $ab_1, b_2, \dots, b_p$  are linearly dependent in  $(\mathcal{O}/\mathfrak{g})^t$ . Then there are  $a_1, \dots, a_p \in \mathcal{O}/\mathfrak{g}$  which are not all zero, such that  $a_1 b_1 + \dots + a_p b_p = 0 \in (\mathcal{O}/\mathfrak{g})^t$ . Hence  $a_1 h_1 + \dots + a_p h_p \equiv 0 \pmod{\mathfrak{g}^2}$ . From this we have  $\mathcal{J}(\mathfrak{h}) \subset \mathfrak{g}$ , a contradiction.  $\square$

**Proposition 10.** *Let  $\mathfrak{h} \subset \mathfrak{g}$  be complete intersection ideals of a polynomial ring  $\mathcal{O}$  over a field  $k$  of characteristic zero. Assume that  $\text{Spec}(\mathcal{O}/\mathfrak{g})$  is reduced and connected, and the Jacobian  $\mathcal{J}(\mathfrak{h})$  is not contained in any minimal prime of  $\mathcal{O}/\mathfrak{g}$ . If in (4.1) we have  $t = p = \text{grade } \mathfrak{h}$ , then  $b := \det(b_{ij})$  is a non-zero divisor in  $\mathcal{O}/\mathfrak{g}$ , and*

- 1) *the images  $\hat{g}_1, \dots, \hat{g}_p$  of  $g_1, \dots, g_p$  generate  $T(M)$  over  $\mathcal{O}/\mathfrak{g}$ ;*
- 2) *the images  $\hat{g}_{p+1}, \dots, \hat{g}_n$  of  $g_{p+1}, \dots, g_n$  generate  $N$  freely over  $\mathcal{O}/\mathfrak{g}$ , so  $M \simeq T(M) \oplus N$ , and  $\text{rank}(M) = \text{rank}(N) = \dim X - \dim \Sigma = n - p$ ;*
- 3) *For each  $\mathfrak{p} \in \text{Ass}(\mathcal{O}/\mathfrak{g})$ , the images  $\tilde{g}_{p+1}, \dots, \tilde{g}_n$  of  $g_{p+1}, \dots, g_n$  form an  $(\mathcal{O}/\mathfrak{h})_{\bar{\mathfrak{p}}}$ -regular sequence and generate  $\bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}$ ;*
- 4)  $\int_{\mathfrak{h}} \mathfrak{g} = (g_1, \dots, g_p) + (g_{p+1}, \dots, g_n)^2$ ;
- 5) *there is a length formula if it is finite*

$$\lambda(\mathfrak{h}, \mathfrak{g}) := l_{\mathcal{O}/\mathfrak{g}}(T(M)) = l_{\mathcal{O}/\mathfrak{g}}\left(\frac{\mathcal{O}}{(b) + \mathfrak{g}}\right).$$

We call  $\lambda(\mathfrak{h}, \mathfrak{g})$  the *torsion number* of the pair  $(\mathfrak{h}, \mathfrak{g})$ . When  $\mathfrak{h}$  and  $\mathfrak{g}$  are clear from the context, we write  $\lambda$  for  $\lambda(\mathfrak{h}, \mathfrak{g})$ .

*Proof.* Since  $t = p$  and  $b$  is a non-zero divisor, one can see that  $\hat{g}_1, \dots, \hat{g}_p \in T(M)$  by multiplying  $B^*$  to the both sides of (4.1), where  $B^*$  is the adjoint matrix of  $B$ .

Since  $\hat{g}_1, \dots, \hat{g}_n$  generate  $M$  over  $\mathcal{O}/\mathfrak{g}$  and (3.1) is exact,  $\pi(\hat{g}_{p+1}), \dots, \pi(\hat{g}_n)$  generate  $N$ . If there is a relation:  $\bar{\beta}_{p+1}\pi(\hat{g}_{p+1}) + \dots + \bar{\beta}_n\pi(\hat{g}_n) = 0 \in N$ , then  $\bar{\beta}_{p+1}\hat{g}_{p+1} + \dots + \bar{\beta}_n\hat{g}_n \in T(M)$ . This means that there is a non-zero divisor  $\bar{\beta} \in \mathcal{O}/\mathfrak{g}$  such that  $\bar{\beta}(\bar{\beta}_{p+1}\hat{g}_{p+1} + \dots + \bar{\beta}_n\hat{g}_n) = 0 \in M$ . By taking representatives, this simply means  $\beta\beta_{p+1}g_{p+1} + \dots + \beta\beta_n g_n \in \mathfrak{g}^2 + \mathfrak{h}$ . Hence there are  $\mu_1, \dots, \mu_p \in \mathcal{O}$  such that

$$\mu_1 h_1 + \dots + \mu_p h_p + \beta\beta_{p+1}g_{p+1} + \dots + \beta\beta_n g_n \in \mathfrak{g}^2.$$

By (4.1), this becomes

$$\mu'_1 g_1 + \dots + \mu'_p g_p + \beta\beta_{p+1}g_{p+1} + \dots + \beta\beta_n g_n \in \mathfrak{g}^2,$$

where  $(\mu'_1 \dots \mu'_p) = (\mu_1 \dots \mu_p) B$ . Since  $g_1, \dots, g_n$  form an  $\mathcal{O}$ -regular sequence, we have  $\bar{\beta}\bar{\beta}_j = 0$  in  $\mathcal{O}/\mathfrak{g}$ . Note that  $\bar{\beta}$  is a non-zero divisor, hence  $\bar{\beta}_j = 0$  in  $\mathcal{O}/\mathfrak{g}$ . This proves 1) and 2).

For each prime  $\mathfrak{p} \in \text{Ass}(\mathcal{O}/\mathfrak{g})$ ,  $N_{\bar{\mathfrak{p}}}$  is also free with the images of  $\hat{g}_{p+1}, \dots, \hat{g}_n$  as basis. Then  $(\int_0 \bar{\mathfrak{g}})_{\bar{\mathfrak{p}}} = \bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}^2$  since  $\bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}$  is a reduced complete intersection in  $(\mathcal{O}/\mathfrak{h})_{\bar{\mathfrak{p}}}$  and  $\mathfrak{p}$  is in the regular locus of  $\mathcal{O}/\mathfrak{h}$  by the assumption. Since  $(\mathcal{O}/\mathfrak{h})_{\bar{\mathfrak{p}}}$  is regular, by Vasconcelos' Theorem [15],  $\tilde{g}_{p+1}, \dots, \tilde{g}_n$  is an  $(\mathcal{O}/\mathfrak{h})_{\bar{\mathfrak{p}}}$ -regular sequence, and they generated  $\bar{\mathfrak{g}}_{\bar{\mathfrak{p}}}$  by Nakayama lemma;

4) follows from Proposition 6.

For the length formula, note that

$$T(M) = \frac{\int_{\mathfrak{h}} \mathfrak{g}}{\mathfrak{g}^2 + \mathfrak{h}} \cong \frac{(g_1, \dots, g_p)}{(g_1, \dots, g_p)^2 + (g_1, \dots, g_p)(g_{p+1}, \dots, g_n) + (h_1, \dots, h_p)}.$$

It is easy to see that

$$M_1 := \frac{(g_1, \dots, g_p)}{(g_1, \dots, g_p)^2 + (g_1, \dots, g_p)(g_{p+1}, \dots, g_n)}$$

is a free  $\mathcal{O}/\mathfrak{g}$ -module. Since  $b$  is a non-zero divisor, the following sequence is exact

$$0 \longrightarrow M_1 \xrightarrow{\phi_B} M_1 \longrightarrow T(M) \longrightarrow 0,$$

where  $\phi_B(\bar{g}_i) := \sum_{j=1}^p \bar{b}_{ij} \bar{g}_j$ . By [2, A.2.6], we have the length formula of  $T(M)$ .  $\square$

Note that in the following, we do not assume (4.1).

**Proposition 11.** *Let  $\mathfrak{h} \subset \mathfrak{g}$  be complete intersection ideals of  $\mathcal{O}$ , a polynomial ring over a field  $k$  of characteristic zero. Let  $\text{grade } \mathfrak{h} = p$  and  $\text{grade } \mathfrak{g} = n$ . Assume that  $\text{Spec}(\mathcal{O}/\mathfrak{g})$  is reduced and connected, and  $\mathcal{J}(\mathfrak{h})$  is not contained in any minimal prime of  $\mathcal{O}/\mathfrak{g}$ . If  $N$  is a free  $\mathcal{O}/\mathfrak{g}$ -module, then*

- 1) *there exists an  $\mathcal{O}$ -regular sequence  $g_1, \dots, g_n$ , generating  $\mathfrak{g}$ , such that*
  - *the images  $\hat{g}_1, \dots, \hat{g}_p$  of  $g_1, \dots, g_p$  generate  $T(M)$ ;*
  - *the images  $\hat{g}_{p+1}, \dots, \hat{g}_n$  of  $g_{p+1}, \dots, g_n$  form a basis of  $N$ ;*
  - *$\text{rank}(M) = \text{rank}(N) = n - p = \dim X - \dim \Sigma$ ;*
- 2)  $\int_{\mathfrak{h}} \mathfrak{g} = (g_1, \dots, g_p) + (g_{p+1}, \dots, g_n)^2$ .
- 3) *we can choose the generators  $h_1, \dots, h_p$  of  $\mathfrak{h}$  such (4.1) holds with  $t = p$  and  $b$  a non-zero divisor in  $\mathcal{O}/\mathfrak{g}$ ;*

*Proof.* Let the images  $\hat{g}_{t+1}, \dots, \hat{g}_n$  of  $g_{t+1}, \dots, g_n \in \mathfrak{g}$  generate  $N$  over  $\mathcal{O}/\mathfrak{g}$ , where  $t := n - \text{rank } N$ .

For any  $\mathfrak{p} \in \text{Ass}(\mathcal{O}/\mathfrak{g})$ , by the assumption,  $\mathfrak{p}$  is in the regular locus of  $\mathcal{O}/\mathfrak{h}$ , and  $N_{\mathfrak{p}}$  is again a free module with the images of  $\hat{g}_{t+1}, \dots, \hat{g}_n$  as basis. By Vasconcelos' theorem (cf. [15]), the images  $\tilde{g}_{t+1}, \dots, \tilde{g}_n$  of  $g_{t+1}, \dots, g_n$  in  $(\mathcal{O}/\mathfrak{h})_{\mathfrak{p}}$  form an  $(\mathcal{O}/\mathfrak{h})_{\mathfrak{p}}$ -regular sequence. And

$$\bar{\mathfrak{g}}_{\mathfrak{p}} = (\tilde{g}_{t+1}, \dots, \tilde{g}_n) + \left( \frac{\int_{\mathfrak{h}} \mathfrak{g}}{\mathfrak{h}} \right)_{\mathfrak{p}} \quad (4.2)$$

Hence the images of  $h_1, \dots, h_p, g_{t+1}, \dots, g_n$  in  $\mathcal{O}_{\mathfrak{p}}$  form an  $\mathcal{O}_{\mathfrak{p}}$ -regular sequence, where  $h_1, \dots, h_p$  form a minimal generating set of  $\mathfrak{h}$ . However, since  $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{p}$ , we have  $n - t + p = \text{grade}(h_1, \dots, h_p, g_{t+1}, \dots, g_n)_{\mathfrak{p}} \leq \text{grade}(\mathfrak{g})_{\mathfrak{p}} = n$ . Hence  $t \geq p$ .

Extend  $g_{t+1}, \dots, g_n$  to an  $\mathcal{O}$ -regular sequence:  $g_1, \dots, g_n$ , such that they generate  $\mathfrak{g}$ . Then  $\hat{g}_1, \dots, \hat{g}_n$  generate  $M$  over  $\mathcal{O}/\mathfrak{g}$ . We look for the generating set of  $T(M)$ . Let

$$\pi(\hat{g}_i) = \bar{c}_{it+1} \pi(\hat{g}_{t+1}) + \dots + \bar{c}_{in} \pi(\hat{g}_n), \quad i = 1, \dots, t.$$

Hence by (3.1),  $\hat{g}'_i := -\hat{g}_i + \bar{c}_{it+1} \hat{g}_{t+1} + \dots + \bar{c}_{in} \hat{g}_n \in T(M)$ ,  $i = 1, \dots, t$ . Taking representatives, denote  $g'_i := -g_i + c_{it+1} g_{t+1} + \dots + c_{in} g_n$ ,  $i = 1, \dots, t$ ,  $g'_{t+j} = g_{t+j}$ ,  $j = 1, \dots, n - t$ . Then  $\mathfrak{g} = (g'_1, \dots, g'_n)$ , with  $\hat{g}'_1, \dots, \hat{g}'_t \in T(M)$ . By Lemma 8,  $t \leq p$ . We have proved 1).

Note that  $\bar{\mathfrak{g}}_{\mathfrak{p}}$  is also a complete intersection ideal in the regular ring  $(\mathcal{O}/\mathfrak{h})_{\mathfrak{p}}$ , and we have  $p = t$  in (4.2). By actually [13] and [4],  $\left( \frac{\int_{\mathfrak{h}} \mathfrak{g}}{\mathfrak{h}} \right)_{\mathfrak{p}} = \bar{\mathfrak{g}}_{\mathfrak{p}}^2$ . By Nakayama lemma and (4.2), we have  $\bar{\mathfrak{g}}_{\mathfrak{p}} = (\tilde{g}_{p+1}, \dots, \tilde{g}_n)$ . By Proposition 6, we have 2).

Since  $\mathfrak{h} \subset \mathfrak{g}$ , we have  $h_i = b_{i1} g'_1 + \dots + b_{ip} g'_p + b_{ip+1} g'_{p+1} + \dots + b_{in} g'_n$ ,  $i = 1, \dots, p$ . For any  $\xi \in \text{Der}_{\mathfrak{h}}(\mathcal{O})$ , we have  $b_{ip+1} \xi(g'_{p+1}) + \dots + b_{in} \xi(g'_n) \equiv 0 \pmod{\mathfrak{g}}$ ,  $i = 1, \dots, p$ . Hence  $b_{ip+1} g'_{p+1} + \dots + b_{in} g'_n \in \int_{\mathfrak{h}} \mathfrak{g}$ , which implies that  $b_{ip+1}, \dots, b_{in} \in \mathfrak{g}$  for  $i = 1, \dots, p$ . It is obvious that  $b$  is a non-zero divisor in  $\mathcal{O}/\mathfrak{g}$ .  $\square$

*Remark 12.* Combining the conclusions in Corollary 7, Proposition 10 and 11, one sees that the Main Theorem is proved. Moreover, either of the equivalent conditions in the Main Theorem is equivalent to 3) in Proposition 11.

*Example 13.* Let  $\mathfrak{h}$  be defined by  $h := x^3 + xy^3 + 2x^2z + 2z^2 = 0$ ,  $\mathfrak{g}$  be defined by  $g_1 := x^2 + y^3 = 0, g_2 := z = 0$ . Thus  $\mathfrak{h} = (h), \mathfrak{g} = (g_1, g_2)$ . Notice that  $h$  is not weighted homogeneous. So it is not easy to find the generator set of  $\text{Der}_{\mathfrak{h}}(\mathcal{O})$ . Then we have the same problem for  $\int_{\mathfrak{h}} \mathfrak{g}$ . If we denote  $g'_1 = g_1 + 2xg_2 + g_2^2$ , then  $h = xg'_1 + (2-x)g_2^2$ , where  $x$  is a non-zero divisor in  $\mathcal{O}/\mathfrak{g}$ . By Proposition 10, we have:

- $T(M)$  is generated by  $g'_1$  over  $\mathcal{O}/\mathfrak{g}$
- $\int_{\mathfrak{h}} \mathfrak{g} = (g'_1, g_2^2) = (x^2 + y^3 + 2xz, z^2)$
- $N = (g_2)/(g'_1g_2, g_2^2)$  is a free  $\mathcal{O}/\mathfrak{g}$ -module.

The following example shows that it is not necessary for  $T(M)$  to be generated by  $\bar{g}_i$  when  $t > p$ .

*Example 14.* Let  $\mathfrak{g} = (g_1, g_2)$  with  $g_1 = xy, g_2 = z$  and  $\mathfrak{h} = (h)$  with  $h = x^2y + yz + z^2 = xg_1 + yg_2 + g_2^2$ . Then  $\mathcal{O}/\mathfrak{g} \cong \mathbb{C}\{x, y\}/(xy)$ ,  $\int_{\mathfrak{h}} \mathfrak{g} = (x^2y, yz, z^2)$  (see Example 17 for this formula) and  $\mathfrak{g}^2 + \mathfrak{h} = (x^2y^2, xyz, z^2, x^2y + yz)$ . So  $T(M) \cong \mathbb{C}x^2y$ . And  $N$  is not a free  $\mathcal{O}/\mathfrak{g}$ -module.

## 5. LINES ON SPACES WITH ISOLATED COMPLETE INTERSECTION SINGULARITIES

We include some applications of the theory to lines on a variety with isolated complete intersection singularity. Let  $\mathcal{O}$  be the polynomial ring  $\mathbb{C}[x, y_1, \dots, y_n]$  or the convergent power series  $\mathbb{C}\{x, y_1, \dots, y_n\}$ . Let  $\Sigma$  be a line in  $\mathbb{C}^{n+1}$  defined by  $\mathfrak{g}$ . Define  ${}_{\Sigma}\mathcal{K} := \mathcal{R}_{\Sigma} \rtimes \mathcal{C}$ , the semi-product of  $\mathcal{R}_{\Sigma}$  with the contact group  $\mathcal{C}$  (cf. [8]), where  $\mathcal{R}_{\Sigma}$  is a subgroup of  $\mathcal{R} := \text{Aut}(\mathcal{O})$  consisting of all the  $\varphi \in \mathcal{R}$  preserving  $\mathfrak{g}$ . This group has an action on the space  $\mathfrak{mg}\mathcal{O}^p$ . For  $h = (h_1, \dots, h_p) \in \mathfrak{mg}\mathcal{O}^p$ , there is an ideal  $\mathfrak{h}$  generated by  $h_1, \dots, h_p$ , and a variety  $X = \mathcal{V}(\mathfrak{h})$ . The image of the differential of  $h: \mathcal{O}^{n+1} \xrightarrow{dh^*} \mathcal{O}^p$  is denoted by  $\text{th}(h)$ . Define a  ${}_{\Sigma}\mathcal{K}$ -invariant

$$\tilde{\lambda} := \tilde{\lambda}(\Sigma, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}^p}{\text{th}(h) + \mathfrak{g}\mathcal{O}^p}.$$

Choose  $\Sigma$  as the  $x$ -axis. Then  $\Sigma$  can be defined by  $\mathfrak{g} = (y_1, \dots, y_n)$ . Denote  $\mathcal{O}_X := \mathcal{O}/\mathfrak{h}, \mathcal{O}_{\Sigma} := \mathcal{O}/\mathfrak{g}$ .

**Proposition 15.** *Let  $\Sigma$  be a line on a variety  $X$  with isolated complete intersection singularity of codimension  $p$ . Then  $h$  is  ${}_{\Sigma}\mathcal{K}$ -equivalent to an  $\tilde{h}$  with components  $\tilde{h}_i \equiv b_i y_i \pmod{\mathfrak{g}^2}$ , where  $b_i \notin \mathfrak{g}, i = 1, \dots, p$ . Moreover  $\lambda(\mathfrak{g}, \mathfrak{h}) = \tilde{\lambda}(\Sigma, X) = \dim_{\mathbb{C}} \mathcal{O}/(b + \mathfrak{g}) = \sum_{k=1}^p l_i$ , where  $b := b_1 \cdots b_p$ , and  $l_i$  is the valuation of  $\bar{b}_i$  in  $\mathcal{O}_{\Sigma}$ .*

*Proof.* Since  $\Sigma \subset X$ , for a given generating set  $\{h_1, \dots, h_p\}$  of  $\mathfrak{h}$ , we have  $h_i \equiv \sum \bar{b}_{ij} y_j \pmod{\mathfrak{g}^2}, i = 1, \dots, p$ , where  $\bar{b}_{ij} \in \mathcal{O}_{\Sigma}$ , and for fixed  $i$ ,  $\bar{b}_{ij}$ 's are not all zero since  $X$  is complete intersection and  $X_{\text{sing}} = \{0\} \subsetneq \Sigma$ . Since  $\mathcal{O}_{\Sigma}$  is a principal ideal domain, by changing the indices, we can assume that  $\bar{b}_{11} \mid \bar{b}_{ij}$ . Let  $y'_1 = y_1 + \sum_{j=2}^n \frac{\bar{b}_{1j}}{\bar{b}_{11}} y_j$ . Then  $h_1 \equiv \bar{b}_{11} y'_1 \pmod{\mathfrak{g}^2}$ . Let  $h'_i = h_i - \frac{\bar{b}_{i1}}{\bar{b}_{11}} h_1, i = 2, \dots, p$ . Repeat the above argument will prove the first part of the proposition.



Consider the exact sequence

$$\mathcal{O}^{n+1} \xrightarrow{dh^*} \mathcal{O}^p \longrightarrow \text{coker}(dh^*) \longrightarrow 0.$$

By tensoring with  $\mathcal{O}_\Sigma$ , we have the exact sequence

$$\mathcal{O}_\Sigma^{n+1} \xrightarrow{d\bar{h}^*} \mathcal{O}_\Sigma^p \longrightarrow \text{coker}(d\bar{h}^*) \longrightarrow 0.$$

However by the expression of  $h_i$ 's above, this is just

$$\mathcal{O}_\Sigma^p \xrightarrow{d\bar{h}^*} \mathcal{O}_\Sigma^p \longrightarrow \frac{\mathcal{O}^p}{\text{th}(h) + \mathfrak{g}\mathcal{O}^p} \longrightarrow 0.$$

Since  $\bar{b} \neq 0$ , by [2, A.2.6], we have the formula for  $\tilde{\lambda}$ . □

**Corollary 16.** *Let  $X$  be a variety with isolated complete intersection singularity of codimension  $p$  in  $\mathbb{C}^{n+1}$ , and  $\Sigma$  a line in  $X$  defined by  $\mathfrak{g}$ . Then we can choose the coordinates of  $\mathbb{C}^{n+1}$  such that  $\mathfrak{g} = (y_1, \dots, y_n)$ ,  $\hat{y}_1, \dots, \hat{y}_p \in T(M)$  and  $\hat{y}_{p+1}, \dots, \hat{y}_n$  generate  $N$  which is free of rank  $n - p$ , and*

$$\int_{\mathfrak{h}} \mathfrak{g} = (y_1, \dots, y_p) + (y_{p+1}, \dots, y_n)^2. \quad \square$$

*Example 17.* Consider the situation in Example 14. Denote  $\mathfrak{g}_1 = (x, z)$  and  $\mathfrak{g}_2 = (y, z)$ . They define  $\Sigma_1 = y$ -axis and  $\Sigma_2 = x$ -axis respectively. We have  $\mathfrak{g} = \mathfrak{g}_1 \cap \mathfrak{g}_2$  and  $\Sigma = \Sigma_1 \cup \Sigma_2 \subset X$ . Since  $h = (y + z)z + yx^2$ , by Corollary 16,  $\int_{\mathfrak{h}} \mathfrak{g}_1 = (x^2, z)$ . Since  $h = (x^2 + z)y + z^2$ , again by Corollary 16,  $\int_{\mathfrak{h}} \mathfrak{g}_2 = (y, z^2)$ . These tell us that  $\int_{\mathfrak{h}} \mathfrak{g} = \int_{\mathfrak{h}} \mathfrak{g}_1 \cap \int_{\mathfrak{h}} \mathfrak{g}_2$ .

**Corollary 18.** (Due to Pellikaan) *Let  $X$  be a variety with isolated complete intersection singularity of codimension  $p$  in  $\mathbb{C}^{n+1}$ , and  $\Sigma$  a line in  $X$ , defined by  $\mathfrak{g} = (y_1, \dots, y_n)$ . Then the Second Exact Sequence [9, 3] is exact on the left also:*

$$0 \longrightarrow M \longrightarrow \Omega_X^1 \otimes \mathcal{O}_\Sigma \longrightarrow \Omega_\Sigma^1 \longrightarrow 0.$$

Furthermore it is splitting and

$$T(\Omega_X^1 \otimes \mathcal{O}_\Sigma) = T(M), \quad \text{rank}(\Omega_X^1 \otimes \mathcal{O}_\Sigma) = n - p - 1.$$

These tell us that the torsion number  $\lambda(\mathfrak{h}, \mathfrak{g})$  is independent of the choice of the generating sets of  $\mathfrak{g}$  and  $\mathfrak{h}$ .

*Proof.* We have the following presentation:

$$\mathcal{O}_X^p \xrightarrow{dh} \mathcal{O}_X^{n+1} \longrightarrow \Omega_X^1 \longrightarrow 0.$$

Tensoring with  $\mathcal{O}_\Sigma$ , we have the exact sequence

$$\mathcal{O}_\Sigma^p \xrightarrow{d\bar{h}} \mathcal{O}_\Sigma^{n+1} \longrightarrow \Omega_X^1 \otimes \mathcal{O}_\Sigma \longrightarrow 0.$$

Remark that the map  $\bar{d}h$  is equivalent to a map defined by the matrix  $(\bar{b}_{ij})$ . Hence

$$\Omega_X^1 \otimes \mathcal{O}_\Sigma \cong \frac{\mathcal{O}_\Sigma^{n+1}}{\text{im} \bar{d}h} \cong \mathcal{O}_\Sigma^{n-p+1} \oplus \frac{\mathcal{O}_\Sigma^p}{\text{im}(\bar{b}_{ij})} \cong \mathcal{O}_\Sigma \oplus N \oplus T(M) \cong \mathcal{O}_\Sigma \oplus M.$$

Then exact is

$$0 \longrightarrow M \longrightarrow \frac{\mathcal{O}_{\Sigma}^{n+1}}{\text{im}dh} \longrightarrow \mathcal{O}_{\Sigma} \longrightarrow 0.$$

Since  $\Omega_{\Sigma}^1$  is free  $\mathcal{O}_{\Sigma}$ -module of rank 1, by [2, A.2.2], we have the exact sequence.  $\square$

*Remark 19.* In general, given an analytic space germ  $(X, 0)$ , one cannot find a smooth curve  $L \subset X$  that passes through and is not contained in  $X_{\text{sing}}$ . However, if there are smooth curves on  $X$  in the above sense, how to distinguish them is a problem. We found that the torsion number  $\lambda$  is a nice candidate for this purpose [7]. In studying the Euler-Poincaré characteristic  $\chi(F)$  of the Milnor fibre  $F$  of a function with singular locus a smooth curve on a singular space, we found that this  $\lambda$  also appears in  $\chi(F)$ . Note also that the torsion number was generalized to “higher torsion numbers” in [6, 7].

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